Systematic derivation of coarse-grained fluctuating hydrodynamic equations for many Brownian particles under nonequilibrium conditions

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We study the statistical properties of many Brownian particles under the influence of both a spatially homogeneous driving force and a periodic potential with period ℓ in a two-dimensional space. In particular, we focus on two asymptotic cases $\ell_{\rm int} \ll \ell$ and $\ell_{\rm int} \gg \ell$, where $\ell_{\rm int}$ represents the interaction length between two particles. We derive fluctuating hydrodynamic equations describing the evolution of a coarse-grained density field defined on scales much larger than ℓ for both cases. Using the obtained equations, we calculate the equal-time correlation functions of the density field to the lowest order of the interaction strength. We find that the system exhibits long-range correlation of the type r^{-d} $(d=2)$ for the case $\ell_{\rm int} \gg \ell$, while no such behavior is observed for the case $\ell_{\rm int} \ll \ell$.

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I. INTRODUCTION

To derive a description of macroscopic phenomena on the basis of a microscopic model is an important problem in statistical physics. In equilibrium systems, equilibrium statistical mechanics provides a clear solution to this problem. However, when a system is out of equilibrium, even in a nonequilibrium steady state, no general framework is known except for linear response theory, which is applied to systems close to equilibrium [1]. Here, let us recall Boltzmann's significant research that led to the genesis of equilibrium statistical mechanics. He arrived at his famous formula by analyzing the simplest system, a dilute gas, thoroughly. Thus, in order to approach nonequilibrium statistical mechanics, we investigate a simple nonequilibrium system by exploring the relation between microscopic and macroscopic descriptions.

Let us consider a system in which many submicrometer particles are driven by an external force in a solvent. The force consists of both a spatially homogeneous driving force and the force generated by a spatially periodic potential. It has been known that such a system can exhibit phenomena out of local equilibrium and that it can be designed for experiments due to the recent development of the optical technology used in controlling and measuring particles [2,3]. The system also provides a typical example of the so-called *driven diffusive system* [4,5]. Theoretically, there exists a case that the motion of the particles is accurately described by a Langevin equation. In this system, we can investigate macroscopic phenomena both by a laboratory experiment and by an analysis of the basis of the Langevin equation that is regarded as a microscopic model. In particular, the systematic calculation of macroscopic quantities from the microscopic model is the first step in developing a new framework of nonequilibrium statistical mechanics.

Macroscopic phenomena in driven diffusive systems have been described phenomenologically within a framework of fluctuating hydrodynamics [5–7]. The dynamical variable in this description is a density field, and its evolution equation is assumed to take the simplest form under the imposed physical requirements. For example, it is assumed that the evolution equation for the density field in driven diffusive systems possesses an anisotropic nature and no detailed balance condition. With this simple assumption, the existence of a long-range correlation in driven diffusive systems was predicted even in a linear model [7,8]. Furthermore, the anomalous behavior of the space-time correlation function of density fluctuations has been studied by analyzing a nonlinear model for driven diffusive systems [9,10].

It is expected that the macroscopic behavior in driven Brownian particle systems is described by a fluctuating hydrodynamic equation. We then address a problem to quantitatively derive the form of fluctuating hydrodynamic equations on the basis of a microscopic model describing the motion of the particles. If this problem is solved, we can calculate the correlation function of density fluctuations by using the obtained fluctuating hydrodynamic equation. Then, the calculation result is more quantitative than that by a fluctuating hydrodynamic equation assumed phenomenologically. In general, it is believed that the equal-time correlation function in d-dimensional driven diffusive systems exhibits the power-law behavior of the type r^{-d} in the long-distance regime and has a short-range part that deviates from the correlation determined by equilibrium statistical mechanics. However, recently, it has been shown that this type of long-range correlation does not appear in some lattice gases [11]. Further, the short-range part of the correlation is connected with an extended thermodynamic function in driven lattice gases [12,13]. Therefore, it is important to calculate a concrete form of the correlation function for driven Brownian particle systems.

Taking these into consideration, in this paper, we derive evolution equations of a coarse-grained density field from a many-body Langevin equation describing the motion of Brownian particles under a nonequilibrium condition. We first note that the many-body Langevin equation is equivalent to a nonlinear fluctuating equation for the density field [14]. To the latter equation, we apply a system reduction method in order to describe the large-scale dynamics of the

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density field [15,16]. As a result, we obtain the evolution equation of the coarse-grained density field and we calculate the equal-time correlation function. Furthermore, with regard to the calculation method, we extend a standard system reduction method so as to analyze a stochastic partial differential equation. Therefore, we expect that this paper contributes to the progress of such perturbation theory, too.

This paper is organized as follows. In Sec. II, we introduce a Langevin equation describing the motion of many Brownian particles under an external force. We then obtain a nonlinear fluctuating hydrodynamic equation for the density field. In Sec. III, we develop a perturbation method to derive a coarse-grained fluctuating hydrodynamic equation and calculate the equal-time correlation functions of density fluctuations. Section IV is devoted to remarks on the present study. The technical details are presented in the Appendixes.

II. MODEL

We study N Brownian particles suspended in a two-dimensional solvent of temperature T. Let $\boldsymbol{x}_i$ be the position of the ith particle. Here, i is an integer satisfying $1 \leqslant i \leqslant N$. We express the αth component of $\boldsymbol{x}_i$ as $x_{i\alpha}$, with $\alpha=1,2$. That is, $\boldsymbol{x}_i=(x_{i1},x_{i2})$. In addition, $\boldsymbol{x}=(x_1,x_2)$ indicates a position in the two-dimensional space, where $0 \leqslant x_{i\alpha} \leqslant L$, and periodic boundary conditions are assumed for simplicity. Each particle is driven by an external force $f\mathbf{e}_1=(f,0)$ and is subject to a periodic potential $U(x_1)$ with period ℓ such that L/ℓ is an integer. For simplicity, we assume that the periodic potential is independent of x_2. Furthermore, we express the interaction between the ith particle and the jth particle by an interaction potential $u(\boldsymbol{x}_i - \boldsymbol{x}_j)$. We assume that the function $u(\boldsymbol{r})$ decays to zero with a typical length $\ell_{\rm int}$. $\bar{\rho}=N/L^2$ represents the average density of the Brownian particles.

The motion of the ith Brownian particle is assumed to be described by a Langevin equation

$$\gamma \frac{dx_{i\alpha}}{dt} = \left(f - \frac{\partial U(x_{i1})}{\partial x_{i\alpha}} \right) \delta_{\alpha 1} - \frac{\partial}{\partial x_{i\alpha}} \sum_{j=1, j\neq i}^{N} u(\boldsymbol{x}_i - \boldsymbol{x}_j) + R_{i\alpha}(t), \tag{1}$$

where γ is a friction constant and $R_{i\alpha}$ is zero-mean Gaussian white noise that satisfies

$$\langle R_{i\alpha}(t) R_{j\beta}(t') \rangle = 2\gamma T \delta_{\alpha\beta} \delta_{ij} \delta(t-t'). \tag{2}$$

It should be noted that without the periodic potential U, the system is equivalent to an equilibrium system in a moving frame with velocity f/γ. Thus, the periodic potential is necessary for investigating the nonequilibrium nature. Note that such a periodic potential can be implemented experimentally by using an optical phase modulator [2,3].

Now, we define the fine-grained density field as

$$\rho_{\rm d}(\boldsymbol{x},t) \equiv \sum_{i=1}^{N} \delta^2(\boldsymbol{x} - \boldsymbol{x}_i(t)). \tag{3}$$

This satisfies the continuity equation

$$\frac{\partial \rho_{\rm d}(\boldsymbol{x},t)}{\partial t} = - \sum_{\alpha=1}^{2} \frac{\partial j_\alpha(\boldsymbol{x},t)}{\partial x_\alpha}, \tag{4}$$

and the expression of current $j_\alpha(\boldsymbol{x},t)$ is derived from Eq. (1) with Eq. (2) as follows:

$$j_\alpha(\boldsymbol{x},t) = \frac{\rho_{\rm d}(\boldsymbol{x},t)}{\gamma} \left[f\delta_{\alpha 1} - \frac{\partial}{\partial x_\alpha} \left(\left. \frac{\delta H}{\delta \varphi(\boldsymbol{x})} \right|_{\varphi(\boldsymbol{x})=\rho_{\rm d}(\boldsymbol{x},t)} \right) \right] + \sqrt{\frac{T\rho_{\rm d}(\boldsymbol{x},t)}{\gamma}}\, \xi_\alpha(\boldsymbol{x},t), \tag{5}$$

where the real-valued functional H for a function $\varphi(\boldsymbol{x})$ takes the form

$$H(\varphi) = \int d^2\boldsymbol{x}\, \varphi(\boldsymbol{x}) U(\boldsymbol{x}) + \frac{1}{2} \int d^2\boldsymbol{x} \int d^2\boldsymbol{y}\, \varphi(\boldsymbol{x}) u(\boldsymbol{x}-\boldsymbol{y}) \varphi(\boldsymbol{y}) + T \int d^2\boldsymbol{x}\, \varphi(\boldsymbol{x}) [\ln \varphi(\boldsymbol{x}) - 1] \tag{6}$$

and $\xi_\alpha(\boldsymbol{x},t)$ in Eq. (5) represents zero-mean space-time Gaussian noise satisfying

$$\langle \xi_\alpha(\boldsymbol{x},t) \xi_\beta(\boldsymbol{x}',t') \rangle = 2\delta_{\alpha\beta} \delta^2(\boldsymbol{x}-\boldsymbol{x}') \delta(t-t'). \tag{7}$$

Throughout this paper, the multiplication of $\xi_\alpha(\boldsymbol{x},t)$ with a usual function $\phi(\boldsymbol{x},t)$ is interpreted as the Stratonovich rule in the space variable and the Ito rule in the time variable. The derivation of fluctuating hydrodynamic equations for the fine-grained density was reported in Ref. [14]. We explain a derivation method of Eq. (4) with Eqs. (5)–(7) in Appendix A.

In the equilibrium case when $f=0$, the form of Eq. (5) is understood by the following physical considerations. The first term in Eq. (5) represents a drift caused by the gradient of the chemical potential that is given by the functional derivative of the potential $H(\rho_{\rm d})$ with respect to $\rho_{\rm d}$. Here, the functional form of $H(\rho_{\rm d})$ may be physically interpreted by noting that the third term in Eq. (6) represents the entropy contribution of noninteracting particles.

The noise term in Eq. (5) can be understood from the fact that the system with $f=0$ satisfies the detailed balance condition. This is verified as follows. Let Δt be a small time interval and $\mathrm{Tr}(\varphi \to \varphi')$ be the conditional probability of the density profile $\varphi'(\boldsymbol{x})$ at time $t+\Delta t$, provided the density profile is $\varphi(\boldsymbol{x})$ at time t. [$\mathrm{Tr}(\varphi \to \varphi')$ is referred to as the transition probability from φ at time t to φ' at time $t+\Delta t$.] For the case $f=0$, using Eqs. (4), (5), and (7), we calculate

$$\mathrm{Tr}(\varphi \to \varphi') = \frac{1}{\mathcal{N}} \exp\left[-\Delta t \int d^2\boldsymbol{x} \frac{\varphi(\boldsymbol{x})}{4\gamma T} \left(\vec{\nabla} \frac{\delta H(\varphi)}{\delta \varphi(\boldsymbol{x})} \right)^2 - \Delta t \int d^2\boldsymbol{x} \frac{\gamma}{4T\varphi(\boldsymbol{x})} \left[\vec{\nabla} \Delta^{-1} \left(\frac{\varphi'(\boldsymbol{x}) - \varphi(\boldsymbol{x})}{\Delta t} \right) \right]^2 - \frac{\Delta t}{2T} \int d^2\boldsymbol{x} \frac{\varphi'(\boldsymbol{x}) - \varphi(\boldsymbol{x})}{\Delta t} \frac{\delta H(\varphi)}{\delta \varphi(\boldsymbol{x})} + O((\Delta t)^2) \right], \tag{8}$$

where Δ^{-1} is the Green function of the Laplacian operator and $\mathcal{N}$ is the normalization constant determined by

$$\int \mathcal{D}\varphi' \mathrm{Tr}(\varphi \to \varphi') = 1, \tag{9}$$

where $\mathcal{D}\phi'$ represents a functional measure. From Eq. (8), we obtain

$$\frac{\mathrm{Tr}(\varphi \to \varphi')}{\mathrm{Tr}(\varphi' \to \varphi)} = \exp\left[-\frac{H(\varphi')-H(\varphi)}{T} + O((\Delta t)^2)\right]. \tag{10}$$

Using Eqs. (9) and (10), we derive

$$\int \mathcal{D}\varphi P_{\mathrm{S}}(\varphi)\mathrm{Tr}(\varphi \to \varphi') = P_{\mathrm{S}}(\varphi'), \tag{11}$$

where

$$P_{\mathrm{S}}(\rho_{\mathrm{d}}) = \frac{1}{Z_{\mathrm{f}}}\exp\left[-\frac{H(\rho_{\mathrm{d}})}{T}\right]. \tag{12}$$

Equation (11) implies that $P_{\mathrm{S}}(\rho_{\mathrm{d}})$ is a steady distribution functional of the density profile ρ_{d}. Then, Eq. (10) is rewritten as

$$P_{\mathrm{S}}(\varphi)\mathrm{Tr}(\varphi \to \varphi') = P_{\mathrm{S}}(\varphi')\mathrm{Tr}(\varphi' \to \varphi)\exp[O((\Delta t)^2)]. \tag{13}$$

This is the detailed balance condition with respect to the distribution given in Eq. (12). It should be noted that ρ_{d} in the third term in Eq. (5) must not be replaced with its average value, because this replacement breaks the detailed balance condition.

In the nonequilibrium case, the effect of the external force f is expressed only by the first term in Eq. (5) as a modification of the gradient of the chemical potential. Although the modification from the equilibrium case is minimum, the modification breaks the detailed balance condition. Therefore, we need to analyze the evolution equation in order to obtain the steady probability distribution of the density field.

III. ANALYSIS

This section consists of four subsections. In Sec. III A, we present the basic framework of our perturbation method. In Sec. III B, we investigate a noninteracting case as the simplest example. We find that there is no long-range correlation of the density field for the noninteracting systems. In Sec. III C, we take into account the effects of the particle interaction perturbatively. In particular, we focus on two asymptotic cases $\ell_{\mathrm{int}} \gg \ell$ and $\ell_{\mathrm{int}} \ll \ell$. In Sec. III D, we calculate the correlation functions of the density field for the two cases and demonstrate that there exists the long-range correlation of the type $1/r^d$ only for the case $\ell_{\mathrm{int}} \gg \ell$.

A. Basic framework

We analyze the nonlinear fluctuating hydrodynamic equation given in Eq. (4) with Eqs. (5)–(7). Our perturbation method is based on the expansion of the weak interaction, weak noise, and separation of length scales. In order to represent the expansion parameters explicitly, we replace $\xi_\alpha(\boldsymbol{x},t)$ and $u(\boldsymbol{x}-\boldsymbol{y})$ with $\mu\xi_\alpha(\boldsymbol{x},t)$ and $\lambda u(\boldsymbol{x}-\boldsymbol{y})$, respectively. We also set $\epsilon \equiv \ell/L$. The parameters μ, ϵ, and λ are regarded as small parameters in our analysis.

With this setting, we first consider the case that $\mu=0$ and $\lambda=0$ in the equation to be analyzed. Because the evolution equation is deterministic in this case, the density field relaxes to the steady one $\rho_{\mathrm{d}}^{(0)}$, which satisfies

$$\bar{\rho}v_{\mathrm{s}} = -\frac{1}{\gamma}\left(\frac{\partial U(x_1)}{\partial x_1} - f + T\frac{\partial}{\partial x_1}\right)\rho_d^{(0)}(\boldsymbol{x}), \tag{14}$$

where v_{s} represents the average velocity of the particle in the steady state, and it is expressed as [17–19]

$$v_{\mathrm{s}} = \frac{T}{\gamma}(1-e^{-\beta f\ell})\left(\int_0^\ell \frac{dx_1}{\ell}I_-(x_1)\right)^{-1}. \tag{15}$$

Here, the function $I_\pm(x_1)$ is defined as

$$I_\pm(x_1) = \int_0^\ell dx_1' \, e^{\pm\beta U(x_1)\mp\beta U(x_1\mp x_1')-\beta f x_1'}. \tag{16}$$

By using the condition

$$\int_0^\ell \frac{dx_1}{\ell}\rho_{\mathrm{d}}^{(0)}(\boldsymbol{x}) = \bar{\rho}, \tag{17}$$

we derive the solution of Eq. (14) as follows:

$$\rho_{\mathrm{d}}^{(0)}(\boldsymbol{x}) = p_{\mathrm{s}}(x_1)\bar{\rho}, \tag{18}$$

where

$$p_{\mathrm{s}}(x_1) = \frac{1}{Z}I_-(x_1). \tag{19}$$

The normalization factor Z is determined by the condition

$$\int_0^\ell dx_1 \, p_{\mathrm{s}}(x_1) = \ell. \tag{20}$$

Now, we consider the case that $\mu \neq 0$ and $\lambda \neq 0$. Based on Eq. (18), we set

$$\rho_{\mathrm{d}}(\boldsymbol{x},t) = p_{\mathrm{s}}(x_1)q(\boldsymbol{x},t). \tag{21}$$

That is, the variable $q(\boldsymbol{x},t)$ is equal to the average density $\bar{\rho}$ when $\mu=0$ and $\lambda=0$, and this variable represents the density modulation caused by the noise and interaction. Substituting Eq. (21) into Eqs. (4), (5), and (6) with the replacement explained in the first paragraph in this subsection, we obtain the evolution equation for $q(\boldsymbol{x},t)$ as

$$\frac{\partial q(\boldsymbol{x},t)}{\partial t} = \hat{M}q(\boldsymbol{x},t) - \mu\frac{1}{p_{\mathrm{s}}(x_1)}\sqrt{\frac{T}{\gamma}}\frac{\partial}{\partial \boldsymbol{x}}\cdot[\sqrt{p_{\mathrm{s}}(x_1)q(\boldsymbol{x},t)}\boldsymbol{\xi}(\boldsymbol{x},t)] - \lambda\frac{1}{p_{\mathrm{s}}(x_1)}\frac{\partial}{\partial \boldsymbol{x}}\cdot\boldsymbol{j}_{\mathrm{int}}(\boldsymbol{x},t), \tag{22}$$

where the operator $\hat{M}$ is calculated as

$$\hat{M} = \left(- \frac{v_s}{p_s(x_1)} + \frac{T}{\gamma} \frac{\partial \ln p_s(x_1)}{\partial x_1} \right) \frac{\partial}{\partial x_1} + \frac{T}{\gamma} \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \right) \tag{23}$$

and $\boldsymbol{j}_{\rm int}(\boldsymbol{x},t)$ is defined as

$$\boldsymbol{j}_{\rm int}(\boldsymbol{x},t) = \frac{1}{\gamma} p_s(x_1) q(\boldsymbol{x},t) \int d^2\boldsymbol{y} \frac{\partial u(\boldsymbol{x}-\boldsymbol{y})}{\partial \boldsymbol{x}} p_s(y_1) q(\boldsymbol{y},t), \tag{24}$$

which represents a current generated by the particle interaction.

Next, we notice the separation of length scales ($\epsilon \ll 1$). We pay attention to the density fluctuations on length scales of the order of L and introduce the density field $Q(\boldsymbol{X},t)$ with a large-scale coordinate $\boldsymbol{X}=\epsilon \boldsymbol{x}$, while on length scales of the order of ℓ the periodic potential determines the system behavior. In order to express this in an explicit manner, we introduce a phase variable θ as $\theta = \mathrm{mod}(x_1, \ell)$ [15,16]. Obviously, $U(\theta)=U(x_1)$ and $p_s(\theta)=p_s(x_1)$.

The density fluctuations with a smaller wave number have a longer time scale. Then, when we focus on a time scale related to diffusion in the entire system, we assume that $Q(\boldsymbol{X},t)$ obeys an autonomous equation and that $q(\boldsymbol{x},t) - Q(\boldsymbol{X},t)$ depends on time t only through the density field $Q(\boldsymbol{X},t)$. These assumptions are expressed as

$$\frac{\partial Q}{\partial t} = \Omega(Q), \tag{25}$$

$$q(\boldsymbol{x},t) = Q(\boldsymbol{X},t) + \rho(\theta,Q), \tag{26}$$

where $\Omega(Q)$ represents a map providing a function of $(\boldsymbol{X},t)$ for the density field Q and $\rho(\theta,Q)$ represents a similar map for each θ (see Refs. [15,16]).

Hereinafter, we treat θ and X as independent variables. Then, when the spatial derivative acts on a function of $(\theta,\boldsymbol{X})$, it should be read as

$$\frac{\partial}{\partial \boldsymbol{x}} = \frac{\partial}{\partial \theta}\mathbf{e}_1 + \epsilon \frac{\partial}{\partial \boldsymbol{X}}. \tag{27}$$

Furthermore, the two quantities $\boldsymbol{\xi}(\boldsymbol{x},t)$ and $u(\boldsymbol{x})$ in Eq. (22) with Eq. (24) can be expressed as $\bar{\boldsymbol{\xi}}(\theta,\boldsymbol{X},t)$ and $\bar{u}(\theta,\boldsymbol{X})$, respectively, by using a method presented in Appendix B. Using this expression, Eq. (7) leads to

$$\langle \bar{\xi}_\alpha(\theta,\boldsymbol{X},t) \bar{\xi}_\beta(\theta',\boldsymbol{X}',t') \rangle = 2\delta_{\alpha\beta} \ell\, \delta(\theta-\theta') \epsilon^2 \delta^2(\boldsymbol{X}-\boldsymbol{X}') \times \delta(t-t'), \tag{28}$$

as explained in Appendix B. Further, using $\bar{u}(\theta,\boldsymbol{X})$, $\boldsymbol{j}_{\rm int}(\boldsymbol{x},t)$ in Eq. (24) is expressed as

$$\bar{\boldsymbol{j}}_{\rm int}(\theta,\boldsymbol{X},t) = p_s(\theta)[Q(\boldsymbol{X},t)+\rho(\theta,Q)] \left(\frac{\partial}{\partial\theta}\mathbf{e}_1 + \epsilon \frac{\partial}{\partial \boldsymbol{X}} \right) \int \frac{d\theta'}{\ell} \int \frac{d^2\boldsymbol{Y}}{\epsilon^2} \bar{u}(\theta-\theta',\boldsymbol{X}-\boldsymbol{Y}) p_s(\theta') \times [Q(\boldsymbol{Y},t)+\rho(\theta',Q)], \tag{29}$$

where we have used Eq. (B9). Finally, $\hat{M}$ can be expanded as

$$\hat{M} = \hat{M}^{(0)} + \epsilon \hat{M}^{(1)} + \epsilon^2 \hat{M}^{(2)}, \tag{30}$$

where $\hat{M}^{(i)}$ with $i=0,1$, and 2 are calculated as

$$\hat{M}^{(0)} = \left(- \frac{v_s}{p_s(\theta)} + \frac{T}{\gamma} \frac{d \ln p_s(\theta)}{d\theta} \right) \frac{\partial}{\partial\theta} + \frac{T}{\gamma} \frac{\partial^2}{\partial\theta^2}, \tag{31}$$

$$\hat{M}^{(1)} = \left(- \frac{v_s}{p_s(\theta)} + \frac{T}{\gamma} \frac{d \ln p_s(\theta)}{d\theta} + 2\frac{T}{\gamma}\frac{\partial}{\partial\theta} \right) \frac{\partial}{\partial X_1}, \tag{32}$$

$$\hat{M}^{(2)} = \frac{T}{\gamma} \left(\frac{\partial^2}{\partial X_1^2} + \frac{\partial^2}{\partial X_2^2} \right). \tag{33}$$

Using these, the substitution of Eqs. (25) and (26) into Eq. (22) yields

$$\Omega(Q) + \frac{\delta\rho}{\delta Q}\cdot\Omega(Q) = \hat{M}[Q+\rho(\theta,Q)] - \mu \frac{1}{p_s(\theta)} \sqrt{\frac{T}{\gamma}} \left(\frac{\partial}{\partial\theta}\mathbf{e}_1 + \epsilon \frac{\partial}{\partial \boldsymbol{X}} \right) \cdot \{ \sqrt{p_s(\theta)[Q+\rho(\theta,Q)]}\, \bar{\boldsymbol{\xi}}(\theta,\boldsymbol{X},t) \} - \lambda \frac{1}{p_s(\theta)} \left(\frac{\partial}{\partial\theta}\mathbf{e}_1 + \epsilon \frac{\partial}{\partial \boldsymbol{X}} \right) \cdot \bar{\boldsymbol{j}}_{\rm int}(\theta,\boldsymbol{X},t), \tag{34}$$

where $(\delta\rho/\delta Q)$ is the operator on a function $\varphi(\boldsymbol{x})$ defined as

$$\rho(\theta,Q+\varphi) - \rho(\theta,Q) = \frac{\delta\rho}{\delta Q}\varphi(\boldsymbol{x}) + O(|\varphi|^2), \tag{35}$$

in the limit $|\varphi| \to 0$ with an appropriate norm $|\cdot|$ of the function space. Recall that ρ is a map in the function space for each θ. Thus, the operator $(\delta\rho/\delta Q)$ mathematically corresponds to a Fréchet derivative when the function space is properly defined. This should not be confused with the functional derivative used in Eq. (5).

Now, we derive $\Omega(Q)$ and $\rho(\theta,Q)$ by solving Eq. (34) with the perturbation method. Because we assumed that ϵ, μ, and λ are small, we expand $\rho(\theta,Q)$ and $\Omega(Q)$ in these parameters. Specifically, setting $\mu=\epsilon$, we assume the form

$$\rho(\theta,Q) = \epsilon\rho_1(\theta,Q) + \epsilon^2\rho_2(\theta,Q) + \cdots, \tag{36}$$

$$\Omega(Q) = \epsilon\Omega_1(Q) + \epsilon^2\Omega_2(Q) + \cdots. \tag{37}$$

Furthermore, for both $\rho_i(\theta,Q)$ and $\Omega_i(Q)$, we consider an expansion with regard to λ. In summary, we analyze Eq. (34) with Eqs. (29), (30), (36), and (37) and $\mu=\epsilon$ and derive $\rho_i(\theta,Q)$ and $\Omega_i(Q)$ iteratively. These calculations are shown in Secs. III B and III C.

In order to make our perturbative calculation concrete, we introduce a space $\mathcal{F}$ consisting of all complex-valued, square-integrable, periodic functions of θ on the interval $[0,\ell]$. We endow this space with the inner product

$$(\alpha,\beta) = \int_0^\ell \frac{d\theta}{\ell}\alpha^*(\theta)\beta(\theta) \tag{38}$$

for $\alpha,\beta \in \mathcal{F}$, where $*$ denotes the complex conjugation. Since the operator $\hat{M}^{(0)}$ is a linear map from $\mathcal{F}$ to $\mathcal{F}$, we can define all the eigenvalues λ_j and the corresponding eigenfunctions $\Phi_j(\theta)$ of the operator $\hat{M}^{(0)}$ in $\mathcal{F}$ by the equation

$$\hat{M}^{(0)}\Phi_j(\theta) = \lambda_j\Phi_j(\theta), \tag{39}$$

where the index $j=0,\pm1,\pm2,\ldots$ is determined by the relations $\lambda_j=\lambda^*_{-j}$ and $\lambda_0=0>\mathrm{Re}(\lambda_{\pm1})>\mathrm{Re}(\lambda_{\pm2})>\cdots$. When a complex eigenvalue is degenerate, the corresponding labeling is modified appropriately. Because $\hat{M}^{(0)}$ is not a Hermitian operator, it is convenient to introduce the adjoint operator of $\hat{M}^{(0)\dagger}$ through the relation

$$(\hat{M}^{(0)\dagger}\alpha,\beta) \equiv (\alpha,\hat{M}^{(0)}\beta). \tag{40}$$

In this space $\mathcal{F}$, $\hat{M}^{(0)\dagger}$ is explicitly represented as

$$\hat{M}^{(0)\dagger} = \frac{\partial}{\partial\theta}\left(\frac{v_{\rm s}}{p_{\rm s}(\theta)} - \frac{T}{\gamma}\frac{d\ln p_{\rm s}(\theta)}{d\theta} + \frac{T}{\gamma}\frac{\partial}{\partial\theta}\right). \tag{41}$$

Note that the set of eigenvalues of $\hat{M}^{(0)\dagger}$ is identical to that of $\hat{M}^{(0)}$. Then, we denote the eigenfunctions of $\hat{M}^{(0)\dagger}$ as $\Psi_j(\theta)$ and choose their labeling such that

$$\hat{M}^{(0)\dagger}\Psi_j(\theta) = \lambda_j^*\Psi_j(\theta). \tag{42}$$

We can choose these eigenfunctions such that the following hold:

$$(\Psi_i,\Phi_j) = \delta_{ij}, \tag{43}$$

$$\sum_{j=-\infty}^{\infty}\frac{\Psi_j^*(\theta)\Phi_j(\theta')}{\ell} = \delta(\theta-\theta'). \tag{44}$$

In particular, we choose the zero eigenfunctions as

$$\Psi_0(\theta) = p_{\rm s}(\theta), \tag{45}$$

$$\Phi_0(\theta) = 1. \tag{46}$$

B. Noninteracting system

We investigate the noninteracting system by setting $\lambda=0$ in Eq. (34) with Eq. (28). Equation (34) can then be rewritten as

$$\Omega + \frac{\delta\rho}{\delta Q}\cdot\Omega(Q) = \hat{M}[Q+\rho(\theta,Q)] - \epsilon\frac{1}{p_{\rm s}(\theta)}\sqrt{\frac{T}{\gamma}}\left(\frac{\partial}{\partial\theta}\mathbf{e}_1 + \epsilon\frac{\partial}{\partial \boldsymbol{X}}\right)\cdot\{\sqrt{p_{\rm s}(\theta)[Q+\rho(\theta,Q)]}\bar{\boldsymbol{\xi}}(\theta,\boldsymbol{X},t)\}, \tag{47}$$

for which we use the expansions given by Eqs. (30), (36), and (37). Selecting all the terms proportional to ϵ in Eq. (47), we obtain

$$\Omega_1(Q) = \hat{M}^{(0)}\rho_1(\theta,Q) + \hat{M}^{(1)}Q - \frac{1}{p_{\rm s}(\theta)}\sqrt{\frac{T}{\gamma}}\frac{\partial}{\partial\theta}[\sqrt{p_{\rm s}(\theta)Q(\boldsymbol{X},t)}\bar{\xi}_1(\theta,\boldsymbol{X},t)]. \tag{48}$$

Here, $\Omega_1(Q)$ and $\rho_1(\theta,Q)$ are unknown functions. In order to derive them, we set

$$s_1(\theta,Q) \equiv \Omega_1(Q) - \hat{M}^{(1)}Q + \frac{1}{p_{\rm s}(\theta)}\sqrt{\frac{T}{\gamma}}\frac{\partial}{\partial\theta}[\sqrt{p_{\rm s}(\theta)Q(\boldsymbol{X},t)}\bar{\xi}_1(\theta,\boldsymbol{X},t)] \tag{49}$$

and rewrite Eq. (48) as

$$\hat{M}^{(0)}\rho_1(\theta,Q) = s_1(\theta,Q). \tag{50}$$

We can regard Eq. (50) as a linear equation with respect to ρ_1 because $s_1(\theta,Q)$ does not contain ρ_1.

Since $\hat{M}^{(0)}$ has a zero eigenvalue, $\hat{M}^{(0)}$ is not invertible. In this case there is no unique solution of ρ_1 to Eq. (50) but either no solution or an infinite number of solutions. Then, in order to perform the perturbative calculation consistently, we impose the solvability condition

$$(\Psi_0,s_1) = 0; \tag{51}$$

under this condition, there exist solutions with an arbitrary constant. This condition determines $\Omega_1(Q)$ as

$$\Omega_1(Q) = (\Psi_0,\hat{M}^{(1)}Q) - \left(\Psi_0,\frac{1}{p_{\rm s}}\sqrt{\frac{T}{\gamma}}\partial[\sqrt{p_{\rm s}Q(\boldsymbol{X},t)}\bar{\xi}_1(\cdot,\boldsymbol{X},t)]\right), \tag{52}$$

where ∂ represents the partial derivative with respect to θ. We note that the second term in the right-hand side of Eq. (52) is equal to zero (see Appendix B). Then, substituting Eq. (32) into Eq. (52), we obtain

$$\Omega_1(Q) = -v_{\rm s}\frac{\partial Q}{\partial X_1}. \tag{53}$$

Under the solvability condition given by Eq. (51), we can derive the following solutions of the linear equation expressed by Eq. (50):

$$\rho_1(\theta,Q) = \sum_{n\neq0}\frac{\Phi_n(\theta)}{-\lambda_n}\left(\Psi_n,-\frac{v_{\rm s}}{p_{\rm s}} + \frac{T}{\gamma}\partial(\ln p_{\rm s})\right)\frac{\partial Q}{\partial X_1} + \sum_{n\neq0}\frac{\Phi_n(\theta)}{\lambda_n}\sqrt{\frac{TQ}{\gamma}}\left(\Psi_n,\frac{1}{p_{\rm s}}\partial[\sqrt{p_{\rm s}}\bar{\xi}_1(\cdot,\boldsymbol{X},t)]\right) + \chi\Phi_0(\theta). \tag{54}$$

Here, χ is an arbitrary constant. We set $\chi=0$ hereafter.

Next, we will determine $\Omega_2(Q)$ and $\rho_2(\theta,Q)$. Using the terms proportional to ϵ^2 in Eq. (47), we obtain

$$\Omega_2(Q) + \frac{\delta\rho_1}{\delta Q}\cdot\Omega_1(Q) = \hat{M}^{(0)}\rho_2 + \hat{M}^{(1)}\rho_1 + \hat{M}^{(2)}Q - \frac{1}{p_s(\theta)}\sqrt{\frac{T}{\gamma}}\frac{\partial}{\partial\theta} \times\left(\sqrt{p_s(\theta)}\frac{\rho_1}{2\sqrt{Q}}\bar{\xi}_1(\theta,\boldsymbol{X},t)\right) - \frac{1}{p_s(\theta)}\sqrt{\frac{T}{\gamma}}\frac{\partial}{\partial\boldsymbol{X}}[\sqrt{p_s(\theta)Q}\bar{\boldsymbol{\xi}}(\theta,\boldsymbol{X},t)]. \tag{55}$$

In the same manner as that in the first-order calculation, we impose the solvability condition for the linear equation of ρ_2. This yields

$$\Omega_2(Q) = (\Psi_0,\hat{M}^{(1)}\rho_1) + (\Psi_0,\hat{M}^{(2)}Q) - \sqrt{\frac{T}{\gamma}}\frac{\partial}{\partial\boldsymbol{X}}\cdot\left[\sqrt{Q}\left(\Psi_0,\frac{1}{\sqrt{p_s}}\bar{\boldsymbol{\xi}}(\cdot,\boldsymbol{X},t)\right)\right]. \tag{56}$$

Using Eqs. (32) and (33), we obtain

$$\Omega_2(Q) = D\frac{\partial^2 Q}{\partial X_1^2} + \frac{T}{\gamma}\frac{\partial^2 Q}{\partial X_2^2} - \frac{\partial}{\partial X_1}\sqrt{Q}[\zeta(\boldsymbol{X},t) + \eta_1(\boldsymbol{X},t)] - \frac{\partial}{\partial X_2}\sqrt{Q}\eta_2(\boldsymbol{X},t). \tag{57}$$

Here, we calculate D, $\zeta(\boldsymbol{X},t)$, and $\eta_\alpha(\boldsymbol{X},t)$ as

$$D = -\left(b,\left[-\frac{v_s}{p_s} + \frac{T}{\gamma}\partial(\ln p_s)\right]\right) + \frac{T}{\gamma}, \tag{58}$$

$$\zeta(\boldsymbol{X},t) = \sqrt{\frac{T}{\gamma}}\left(\partial\left(\frac{b}{p_s}\right),\sqrt{p_s}\bar{\xi}_1(\cdot,\boldsymbol{X},t)\right), \tag{59}$$

$$\eta_\alpha(\boldsymbol{X},t) = \sqrt{\frac{T}{\gamma}}\left(\Psi_0,\frac{1}{\sqrt{p_s}}\bar{\xi}_\alpha(\cdot,\boldsymbol{X},t)\right), \tag{60}$$

where $b(\theta)$ is defined as

$$b(\theta) \equiv \sum_{m\neq 0}\left(\Psi_0,\left[-\frac{v_s}{p_s} + \frac{T}{\gamma}\partial(\ln p_s) + \frac{2T}{\gamma}\partial\right]\Phi_m^*\right)\frac{\Psi_m(\theta)}{\lambda_m^*}. \tag{61}$$

We find that $b(\theta)$ is a real function [see Eq. (C11)].

Now, we write the coarse-grained hydrodynamic equation by defining

$$\tilde{Q}(\boldsymbol{x},t) \equiv Q(\boldsymbol{X},t), \tag{62}$$

$$\Xi_1(\boldsymbol{x},t) \equiv \frac{1}{\epsilon}[\zeta(\boldsymbol{X},t) + \eta_1(\boldsymbol{X},t)], \tag{63}$$

$$\Xi_2(\boldsymbol{x},t) \equiv \frac{1}{\epsilon}\eta_2(\boldsymbol{X},t). \tag{64}$$

Using these and from Eqs. (25), (53), and (57), we obtain

$$\frac{\partial\tilde{Q}}{\partial t} = -\sum_{\alpha=1}^{2}\frac{\partial\tilde{J}_\alpha(\boldsymbol{x},t)}{\partial x_\alpha}, \tag{65}$$

with

$$\tilde{J}_1(\boldsymbol{x},t) = v_s\tilde{Q}(\boldsymbol{x},t) - D\frac{\partial\tilde{Q}}{\partial x_1} + \sqrt{\tilde{Q}(\boldsymbol{x},t)}\Xi_1(\boldsymbol{x},t),$$

$$\tilde{J}_2(\boldsymbol{x},t) = -\frac{T}{\gamma}\frac{\partial\tilde{Q}}{\partial x_2} + \sqrt{\tilde{Q}(\boldsymbol{x},t)}\Xi_2(\boldsymbol{x},t), \tag{66}$$

where Ξ_α with $\alpha=1,2$ satisfies

$$\langle\Xi_\alpha(\boldsymbol{x},t)\Xi_\beta(\boldsymbol{x}',t')\rangle = 2B_{\alpha\beta}\delta^2(\boldsymbol{x}-\boldsymbol{x}')\delta(t-t'). \tag{67}$$

The noise intensities $B_{\alpha\beta}$ are calculated as $B_{12}=B_{21}=0$ and

$$B_{11} = \frac{T}{\gamma}\int_0^\ell\frac{d\theta}{\ell}p_s(\theta)\left[\frac{d}{d\theta}\left(\frac{b(\theta)}{p_s(\theta)}\right)+1\right]^2, \tag{68}$$

$$B_{22} = \frac{T}{\gamma}. \tag{69}$$

See Appendix C for the calculation.

Here, we present two remarks on the coarse-grained hydrodynamic equation given by Eq. (65) with Eqs. (66) and (67). The first remark is on the expression of D given by Eq. (58). Although the expression is complicated, we can rewrite Eq. (58) as

$$D = \frac{T}{\gamma}\left(\int_0^\ell\frac{d\theta}{\ell}I_-(\theta)\right)^{-3}\int_0^\ell\frac{d\theta}{\ell}[I_-(\theta)]^2 I_+(\theta) \tag{70}$$

using Eq. (16). The derivation is presented in Appendix C. This expression of the diffusion constant coincides with that of the diffusion constant of a Brownian particle in the tilted periodic potential [17–19]. Physically, this coincidence is obvious, because we consider the non-interacting particles in this subsection.

The second remark is on a special relation

$$B_{11} = D. \tag{71}$$

The proof of this relation is presented in Appendix C. This relation corresponds to the fluctuation-dissipation relation of the second kind in this fluctuating hydrodynamic equation. Using this property, in the same manner as that in Sec. II, we can prove the detailed balance condition of the system in the moving frame with velocity v_s. From this condition, we find the steady probability distribution functional of the coarse-grained density field $P_S(Q)$ as

$$P_S(Q) = \frac{1}{Z_c}\exp\left(-\int d^2\boldsymbol{x}\, Q(\boldsymbol{x})[\ln Q(\boldsymbol{x}) - 1]\right). \tag{72}$$

This implies that in the noninteracting system, the density field does not exhibit a long-range spatial correlation even if the system is out of equilibrium.

C. Weakly interacting system

In this subsection, we extend the analysis in the previous subsection to a system consisting of interacting particles.

Concretely, we expand $\rho_i(\theta,Q)$ and $\Omega_i(Q)$ in Eqs. (36) and (37) as

$$\rho_i(\theta,Q)=\rho_{i0}(\theta,Q)+\lambda\rho_{i1}(\theta,Q)+\lambda^2\rho_{i2}(\theta,Q)+\cdots, \tag{73}$$

$$\Omega_i(Q)=\Omega_{i0}(Q)+\lambda\Omega_{i1}(Q)+\lambda^2\Omega_{i2}(Q)+\cdots. \tag{74}$$

Further, in order to make the calculation results explicit, we consider two asymptotic cases: (i) $\ell_{\rm int}\gg\ell$ and (ii) $\ell_{\rm int}\ll\ell$. By substituting Eqs. (73) and (74) into Eq. (34) with Eq. (24) for the two cases, we determine $\rho_{ik}(\theta,Q)$ and Ω_{ik} iteratively.

1. Case (i)

We study the case $\ell_{\rm int}\gg\ell$. Specifically, we assume that $\ell_{\rm int}\simeq O(\epsilon^{-1}\ell)$, and then we set

$$\bar{u}(\theta,\boldsymbol{X})=\epsilon^2 u_{\rm L}(\boldsymbol{X}). \tag{75}$$

The factor ϵ^2 is introduced in order to develop a systematic perturbation method. Substituting Eq. (75) into Eq. (29), we express the current generated by this interaction potential as

$$\bar{\boldsymbol{j}}_{\rm int}(\theta,\boldsymbol{X},t)=\frac{1}{\gamma}p_{\rm s}(\theta)[Q(\boldsymbol{X},t)+\rho(\theta,Q)]\int d^2\boldsymbol{Y}\,\epsilon\frac{\partial u_{\rm L}(\boldsymbol{X}-\boldsymbol{Y})}{\partial \boldsymbol{X}}Q(\boldsymbol{Y},t). \tag{76}$$

Then, Eq. (34) with Eq. (29) becomes

$$\begin{aligned}\Omega(Q)+\frac{\delta\rho}{\delta Q}\cdot\Omega(Q)&=\hat{M}[Q+\rho(\theta,Q)]-\epsilon\frac{1}{p_{\rm s}(\theta)}\sqrt{\frac{T}{\gamma}}\left(\frac{\partial}{\partial\theta}\mathbf{e}_1+\epsilon\frac{\partial}{\partial\boldsymbol{X}}\right)\cdot\{\sqrt{p_{\rm s}(\theta)[Q+\rho(\theta,Q)]}\bar{\boldsymbol{\xi}}(\theta,\boldsymbol{X},t)\}\\&\quad-\frac{\lambda}{\gamma}\epsilon\frac{1}{p_{\rm s}(\theta)}\left(\frac{\partial}{\partial\theta}\mathbf{e}_1+\epsilon\frac{\partial}{\partial\boldsymbol{X}}\right)\cdot\left\{p_{\rm s}(\theta)[Q+\rho(\theta,Q)]\int d^2\boldsymbol{Y}\frac{\partial u_{\rm L}(\boldsymbol{X}-\boldsymbol{Y})}{\partial\boldsymbol{X}}Q(\boldsymbol{Y},t)\right\}.\end{aligned} \tag{77}$$

We now analyze Eq. (77) with the expansions given by Eqs. (30), (36), (37), (73), and (74).

Selecting all the terms proportional to ϵ, we obtain

$$\begin{aligned}\Omega_1(Q)&=\hat{M}^{(0)}\rho_1+\hat{M}^{(1)}Q-\frac{1}{p_{\rm s}(\theta)}\sqrt{\frac{T}{\gamma}}\frac{\partial}{\partial\theta}[\sqrt{p_{\rm s}(\theta)Q}\bar{\xi}_1(\theta,\boldsymbol{X},t)]\\&\quad-\frac{\lambda}{\gamma}\frac{1}{p_{\rm s}(\theta)}\frac{\partial p_{\rm s}(\theta)}{\partial\theta}Q(\boldsymbol{X},t)\int d^2\boldsymbol{Y}\frac{\partial u_{\rm L}(\boldsymbol{X}-\boldsymbol{Y})}{\partial X_1}Q(\boldsymbol{Y},t).\end{aligned} \tag{78}$$

The terms independent of λ reproduce Eq. (48). Thus, $\Omega_{10}(Q)$ and ρ_{10} are equal to $\Omega_1(Q)$ and ρ_1 given by Eqs. (53) and (54), respectively. Next, extracting all the terms proportional to λ in Eq. (78), we obtain

$$\hat{M}^{(0)}\rho_{11}=\Omega_{11}(Q)+\frac{1}{\gamma}\frac{1}{p_{\rm s}(\theta)}\frac{\partial p_{\rm s}(\theta)}{\partial\theta}\times Q(\boldsymbol{X},t)\int d^2\boldsymbol{Y}\frac{\partial u_{\rm L}(\boldsymbol{X}-\boldsymbol{Y})}{\partial X_1}Q(\boldsymbol{Y},t). \tag{79}$$

The solvability condition for the linear equation of ρ_{11} yields

$$\Omega_{11}(Q)=0. \tag{80}$$

Then, we can solve ρ_{11} as

$$\rho_{11}(\theta,Q)=\frac{1}{\gamma}\sum_{n\neq0}\frac{\Phi_n(\theta)}{\lambda_n}\left(\Psi_n,\frac{1}{p_{\rm s}}\partial p_{\rm s}\right)\times Q(\boldsymbol{X},t)\int d^2\boldsymbol{Y}\frac{\partial u_{\rm L}(\boldsymbol{X}-\boldsymbol{Y})}{\partial X_1}Q(\boldsymbol{Y},t), \tag{81}$$

where the term proportional to $\Phi_0(\theta)$ is set to zero.

Next, using all the terms proportional to ϵ^2 in Eq. (77) with the expansions, we calculate $\Omega_{20}(Q)$ and $\Omega_{21}(Q)$ by repeating the same analysis. Obviously, $\Omega_{20}(Q)$ is equal to $\Omega_2(Q)$ in Eq. (57). Extracting the term proportional to $\lambda\epsilon^2$ in Eq. (77), we obtain

$$\begin{aligned}\Omega_{21}(Q)+\frac{\delta\rho_{11}}{\delta Q}\cdot\Omega_{10}(Q)+\frac{\delta\rho_{10}}{\delta Q}\cdot\Omega_{11}(Q)&=\hat{M}^{(0)}\rho_{21}+\hat{M}^{(1)}\rho_{11}\\&\quad-\frac{1}{p_{\rm s}(\theta)}\sqrt{\frac{T}{\gamma}}\frac{\partial}{\partial\theta}\left[\sqrt{p_{\rm s}(\theta)}\frac{\rho_{11}(\theta,Q)}{2\sqrt{Q(\boldsymbol{X},t)}}\bar{\xi}_1(\theta,\boldsymbol{X},t)\right]\\&\quad-\frac{1}{\gamma}\frac{1}{p_{\rm s}(\theta)}\frac{\partial}{\partial\theta}[p_{\rm s}(\theta)\rho_{10}(\theta,Q)]\int d^2\boldsymbol{Y}\frac{\partial u_{\rm L}(\boldsymbol{X}-\boldsymbol{Y})}{\partial X_1}Q(\boldsymbol{Y},t)\\&\quad-\frac{1}{\gamma}\frac{\partial}{\partial\boldsymbol{X}}\left[Q(\boldsymbol{X},t)\int d^2\boldsymbol{Y}\frac{\partial u_{\rm L}(\boldsymbol{X}-\boldsymbol{Y})}{\partial\boldsymbol{X}}Q(\boldsymbol{Y},t)\right].\end{aligned} \tag{82}$$

Then, the solvability condition for the linear equation of ρ_{21} yields

$$\Omega_{21}(Q)=(\Psi_0,\hat{M}^{(1)}\rho_{11})-\frac{1}{\gamma}\frac{\partial}{\partial\boldsymbol{X}}\cdot\left(Q(\boldsymbol{X},t)\int d^2\boldsymbol{Y}\frac{\partial u_{\rm L}(\boldsymbol{X}-\boldsymbol{Y})}{\partial\boldsymbol{X}}Q(\boldsymbol{Y},t)\right). \tag{83}$$

Substituting Eq. (81) into the first term on the right-hand side of Eq. (83) leads to

$$(\Psi_0,\hat{M}^{(1)}\rho_{11})=\frac{1}{\gamma}\left(b,\frac{1}{p_{\rm s}}\partial p_{\rm s}\right)\frac{\partial}{\partial X_1}\times\left(Q(\boldsymbol{X},t)\int d^2\boldsymbol{Y}\frac{\partial u_{\rm L}(\boldsymbol{X}-\boldsymbol{Y})}{\partial X_1}Q(\boldsymbol{Y},t)\right), \tag{84}$$

where $b(\theta)$ is given by Eq. (61). Here, we note the identity

$$\left(b,\frac{1}{p_s}\partial p_s\right)-1=-\int_0^\ell \frac{d\theta}{\ell}p_s(\theta)\left(\frac{\partial}{\partial\theta}\left[\frac{b(\theta)}{p_s(\theta)}\right]+1\right)$$
$$=-\left(\int_0^\ell \frac{d\theta'}{\ell}I_-(\theta')\right)^{-2}\int_0^\ell \frac{d\theta}{\ell}I_-(\theta)I_+(\theta)$$
$$=-\gamma\frac{dv_s}{df}, \qquad (85)$$

where we have used Eq. (C1) to obtain the third line and the fourth line can be confirmed directly from Eq. (15) (see also Ref. [19]). Using Eqs. (84) and (85), we rewrite Eq. (83) as

$$\Omega_{21}(Q)=-\frac{dv_s}{df}\frac{\partial}{\partial X_1}\left[Q(\boldsymbol{X},t)\int d^2\boldsymbol{Y}\frac{\partial u_L(\boldsymbol{X}-\boldsymbol{Y})}{\partial X_1}Q(\boldsymbol{Y},t)\right]-\frac{1}{\gamma}\frac{\partial}{\partial X_2}\left[Q(\boldsymbol{X},t)\int d^2\boldsymbol{Y}\frac{\partial u_L(\boldsymbol{X}-\boldsymbol{Y})}{\partial X_2}Q(\boldsymbol{Y},t)\right]. \qquad (86)$$

Finally, we derive the coarse-grained hydrodynamic equation from Eqs. (25), (53), (57), (80), and (86). Using the variable defined in Eqs. (62), (63), and (64) we obtain the continuity equation for $\tilde{Q}(\boldsymbol{x},t)$ expressed by Eq. (65) with the current as follows:

$$\tilde{J}_1(\boldsymbol{x},t)=v_s\tilde{Q}(\boldsymbol{x},t)-D\frac{\partial\tilde{Q}}{\partial x_1}+\sqrt{\tilde{Q}(\boldsymbol{x},t)}\Xi_1(\boldsymbol{x},t)+\lambda\frac{D}{T}(1-\delta)\tilde{Q}(\boldsymbol{x},t)\int d^2\boldsymbol{y}\frac{\partial u(\boldsymbol{x}-\boldsymbol{y})}{\partial x_1}\tilde{Q}(\boldsymbol{y},t),$$

$$\tilde{J}_2(\boldsymbol{x},t)=-\frac{T}{\gamma}\frac{\partial\tilde{Q}}{\partial x_2}+\sqrt{\tilde{Q}(\boldsymbol{x},t)}\Xi_2(\boldsymbol{x},t)+\lambda\frac{1}{\gamma}\tilde{Q}(\boldsymbol{x},t)\int d^2\boldsymbol{y}\frac{\partial u(\boldsymbol{x}-\boldsymbol{y})}{\partial x_2}\tilde{Q}(\boldsymbol{y},t), \qquad (87)$$

where δ is the dimensionless parameter defined as

$$\delta\equiv 1-\frac{T}{D}\frac{dv_s}{df}. \qquad (88)$$

The Einstein relation leads to $\delta=0$ when $f=0$, while $\delta\neq 0$ when $f\neq 0$ as far as we checked numerically (see Ref. [19]).

When $\delta=0$, by using a similar argument as that in Sec. II, we can prove the detailed balance condition of the system in the moving frame with velocity v_s. However, when $\delta\neq 0$, the coarse-grained hydrodynamic equation does not possess the detailed balance property for any moving frame because we cannot construct a potential function such as H for the argument in Sec. II. It should be noted that the noise intensities are not modified by the lowest order contribution of the interaction effects. Therefore, the fluctuation-dissipation relation of the second kind is maintained in this hydrodynamic equation.

2. Case (ii)

Next, we study the case $\ell_{\rm int}\ll\ell$. We assume the form

$$\bar{u}(\theta,\boldsymbol{X})=u_0\ell\,\delta(\theta)\epsilon^2\delta^2(\boldsymbol{X}), \qquad (89)$$

where the intensity u_0 is determined from the interaction potential with $\ell_{\rm int}\ll\ell$. Here, we set $u_0=\epsilon u_S$ in order to develop a systematic perturbation. Note that the quantity u_0 should appear when we calculate experimentally measurable quantities. Substituting Eq. (89) into Eq. (29), we obtain

$$\bar{\boldsymbol{j}}_{\rm int}(\theta,\boldsymbol{X},t)=\epsilon\frac{u_S}{\gamma}p_s(\theta)[Q+\rho(\theta,Q)]\left(\frac{\partial}{\partial\theta}\mathbf{e}_1+\epsilon\frac{\partial}{\partial\boldsymbol{X}}\right)\{p_s(\theta)[Q+\rho(\theta,Q)]\}. \qquad (90)$$

Then, Eq. (34) with Eq. (29) becomes

$$\Omega(Q)+\frac{\delta\rho}{\delta Q}\cdot\Omega(Q)=\hat{M}[Q+\rho(\theta,Q)]-\mu\sqrt{\frac{T}{\gamma}}\frac{1}{p_s(\theta)}\left(\frac{\partial}{\partial\theta}\mathbf{e}_1+\epsilon\frac{\partial}{\partial\boldsymbol{X}}\right)\cdot[\sqrt{p_s(\theta)[Q+\rho(\theta,Q)]}\bar{\boldsymbol{\xi}}(\theta,\boldsymbol{X},t)]-\epsilon\frac{\lambda u_S}{\gamma}\frac{1}{p_s(\theta)}\left(\frac{\partial}{\partial\theta}\mathbf{e}_1+\epsilon\frac{\partial}{\partial\boldsymbol{X}}\right)\cdot\left\{p_s(\theta)[Q+\rho(\theta,Q)]\left(\frac{\partial}{\partial\theta}\mathbf{e}_1+\epsilon\frac{\partial}{\partial\boldsymbol{X}}\right)p_s(\theta)[Q+\rho(\theta,Q)]\right\}. \qquad (91)$$

We analyze Eq. (91) using the expansions given in Eqs. (30), (36), (37), (73), and (74).

The calculation procedures hereafter are the same as that for case (i). Thus, without repeating the calculations, we summarize the results as follows:

$$\Omega_{11}(Q)=0, \qquad (92)$$

$$\rho_{11}(\theta,Q)=Q^2\frac{u_S}{\gamma}\sum_{n\neq 0}\frac{\Phi_n(\theta)}{\lambda_n}\times\int_0^\ell\frac{d\theta'}{\ell}\frac{\Psi_n^*(\theta')}{p_s(\theta')}\frac{d}{d\theta'}\left[p_s(\theta')\frac{dp_s(\theta')}{d\theta'}\right], \qquad (93)$$

$$\Omega_{21}(Q)=-\frac{\partial}{\partial X_1}[\bar{\nu}Q^2(\boldsymbol{X},t)], \qquad (94)$$

where $\bar{\nu}$ is defined as

$$\bar{\nu}=\frac{u_S}{\gamma}\left(\int_0^\ell\frac{d\theta}{\ell}I_-(\theta)\right)^{-1}\int_0^\ell\frac{d\theta}{\ell}p_s(\theta)I_+(\theta)\frac{dp_s(\theta)}{d\theta}. \qquad (95)$$

Note that in the equilibrium case ($f=0$), $\bar{\nu}=0$ because $p_s(\theta)I_+(\theta)$ is equal to unity when $f=0$.

Finally, we derive the coarse-grained hydrodynamic equation from Eqs. (53), (57), (92), and (94). Using the variable defined by Eqs. (62), (63), and (64), we obtain the continuity equation for $\tilde{Q}(\boldsymbol{x},t)$ expressed by Eq. (65) with the current as follows:

$$\tilde{J}_1(\boldsymbol{x},t) = v_s\tilde{Q}(\boldsymbol{x},t) - D\frac{\partial\tilde{Q}}{\partial x_1} + \sqrt{\tilde{Q}(\boldsymbol{x},t)}\Xi_1(\boldsymbol{x},t) + \lambda\nu\tilde{Q}^2(\boldsymbol{x},t),$$

$$\tilde{J}_2(\boldsymbol{x},t) = -\frac{T}{\gamma}\frac{\partial\tilde{Q}(\boldsymbol{x},t)}{\partial x_2} + \sqrt{\tilde{Q}(\boldsymbol{x},t)}\Xi_2(\boldsymbol{x},t). \quad (96)$$

Here, we have defined ν as

$$\nu \equiv \frac{u_0}{u_S}\bar{\nu}, \quad (97)$$

where the quantity ν is independent of ϵ and its value can be determined from the Langevin model we study. Note that the coarse-grained hydrodynamic equation does not possess the detailed balance property for any moving frame except for the case where $\nu=0$.

D. Correlation function

In this subsection, we calculate the equal-time correlation function $C_0(\boldsymbol{r})$ of the coarse-grained density field for the two above-mentioned cases, case (i) $\ell_{\mathrm{int}} \gg \ell$ and case (ii) $\ell_{\mathrm{int}} \ll \ell$. We first define the space-time correlation function as

$$C(\boldsymbol{r},\tau) \equiv \langle \psi(\boldsymbol{x},t)\psi(\boldsymbol{x}+\boldsymbol{r},t+\tau)\rangle, \quad (98)$$

where $\psi(\boldsymbol{x},t)$ represents the deviation of the density field from the average value $\bar{\rho}$ as follows:

$$\psi(\boldsymbol{x},t) \equiv \tilde{Q}(\boldsymbol{x},t) - \bar{\rho}. \quad (99)$$

In Eq. (98), we assume that the statistical properties of $\psi(\boldsymbol{x},t)$ are translational invariant in the space and time directions.

Hereinafter, we denote the Fourier transformation of a function $f(\boldsymbol{x},t)$ as

$$\hat{f}(\boldsymbol{k},\omega) = \int d^2\boldsymbol{x}\, dt\, f(\boldsymbol{x},t)e^{-i\boldsymbol{k}\cdot\boldsymbol{x}-i\omega t}. \quad (100)$$

We also use the same notation for the Fourier transformation of a function $f(\boldsymbol{x})$ as follows:

$$\hat{f}(\boldsymbol{k}) = \int d^2\boldsymbol{x}\, f(\boldsymbol{x})e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}. \quad (101)$$

Then, we can derive the relation

$$\hat{C}(\boldsymbol{k},\omega)(2\pi)^3\delta^2(\boldsymbol{k}+\boldsymbol{k}')\delta(\omega+\omega') = \langle\hat{\psi}(\boldsymbol{k},\omega)\hat{\psi}(\boldsymbol{k}',\omega')\rangle. \quad (102)$$

Further, the equal-time correlation function $C_0(\boldsymbol{r})$ is given by

$$C_0(\boldsymbol{r}) = \int \frac{d^2\boldsymbol{k}}{(2\pi)^2}\hat{C}_0(\boldsymbol{k})e^{i\boldsymbol{k}\cdot\boldsymbol{r}} = \int \frac{d^2\boldsymbol{k}}{(2\pi)^2}\frac{d\omega}{2\pi}\hat{C}(\boldsymbol{k},\omega)e^{i\boldsymbol{k}\cdot\boldsymbol{r}}. \quad (103)$$

In the argument below, we first calculate $\hat{C}(\boldsymbol{k},\omega)$ from the coarse-grained hydrodynamic equations for cases (i) and (ii), and we derive $C_0(\boldsymbol{r})$ using Eq. (103).

1. Case (i)

Let us consider the case $\ell_{\mathrm{int}} \gg \ell$. Substituting Eq. (99) into the continuity equation expressed by Eq. (65) with Eq. (87), we can obtain the equation for ψ. We further linearize the obtained equation. Then, the resultant equation becomes

$$\frac{\partial\psi}{\partial t} = -\sum_{\alpha=1}^{2}\frac{\partial\mathcal{J}_\alpha}{\partial x_\alpha}, \quad (104)$$

with the current $\mathcal{J}_\alpha$ expressed as

$$\mathcal{J}_1(\boldsymbol{x},t) = v_s\psi - D\frac{\partial\psi}{\partial x_1} - D\frac{\lambda\bar{\rho}(1-\delta)}{T}\int d^2\boldsymbol{y}\frac{\partial u(\boldsymbol{x}-\boldsymbol{y})}{\partial x_1}\psi(\boldsymbol{y},t) + \sqrt{\bar{\rho}}\Xi_1(\boldsymbol{x},t),$$

$$\mathcal{J}_2(\boldsymbol{x},t) = -\frac{T}{\gamma}\frac{\partial\psi}{\partial x_2} - \frac{\lambda\bar{\rho}}{\gamma}\int d^2\boldsymbol{y}\frac{\partial u(\boldsymbol{x}-\boldsymbol{y})}{\partial x_2}\psi(\boldsymbol{y},t) + \sqrt{\bar{\rho}}\Xi_2(\boldsymbol{x},t). \quad (105)$$

The Fourier transform of the evolution equation is written as

$$\hat{\psi}(\boldsymbol{k},\omega) = G(\boldsymbol{k},\omega)[-i\sqrt{\bar{\rho}}\boldsymbol{k}\cdot\hat{\boldsymbol{\Xi}}(\boldsymbol{k},\omega)]. \quad (106)$$

Here, $G(\boldsymbol{k},\omega)$ is the Green function calculated as

$$\frac{1}{G(\boldsymbol{k},\omega)} = i(\omega + v_s k_1) + g_\delta(\boldsymbol{k})Dk_1^2 + g_0(\boldsymbol{k})\frac{T}{\gamma}k_2^2, \quad (107)$$

where we have defined $g_\delta(\boldsymbol{k})$ as

$$g_\delta(\boldsymbol{k}) \equiv 1 + \frac{\lambda\bar{\rho}\hat{u}(\boldsymbol{k})}{T}(1-\delta). \quad (108)$$

Using the relation

$$\langle\hat{\Xi}_\alpha(\boldsymbol{k},\omega)\hat{\Xi}_\beta(\boldsymbol{k}',\omega')\rangle = (2\pi)^3 2B_{\alpha\beta}\delta^2(\boldsymbol{k}+\boldsymbol{k}')\delta(\omega+\omega'), \quad (109)$$

we obtain

$$\hat{C}(\boldsymbol{k},\omega) = 2\bar{\rho}|G(\boldsymbol{k},\omega)|^2\left(Dk_1^2 + \frac{T}{\gamma}k_2^2\right). \quad (110)$$

Integrating Eq. (110) with Eq. (107) over the frequency, we calculate

$$\hat{C}_0(\boldsymbol{k}) = \bar{\rho}\frac{g_{\delta/2}(\boldsymbol{k})}{g_\delta(\boldsymbol{k})g_0(\boldsymbol{k})} + \lambda\bar{\rho}\delta\frac{\bar{\rho}\hat{u}(\boldsymbol{k})}{2T}\frac{1}{g_\delta(\boldsymbol{k})g_0(\boldsymbol{k})} \times \frac{g_\delta(\boldsymbol{k})Dk_1^2 - g_0(\boldsymbol{k})(T/\gamma)k_2^2}{g_\delta(\boldsymbol{k})Dk_1^2 + g_0(\boldsymbol{k})(T/\gamma)k_2^2}. \quad (111)$$

The asymptotic behavior in the range $|\boldsymbol{k}| \ll \ell_{\mathrm{int}}^{-1}$ in Eq. (111) is evaluated as

$$\hat{C}_0(\boldsymbol{k}) \simeq \bar{\rho}\frac{g_{\delta/2}(0)}{g_\delta(0)g_0(0)} + \lambda\bar{\rho}\delta\frac{\bar{\rho}\hat{u}(0)}{2T}\frac{1}{g_\delta(0)g_0(0)} \times \frac{g_\delta(0)Dk_1^2 - g_0(0)(T/\gamma)k_2^2}{g_\delta(0)Dk_1^2 + g_0(0)(T/\gamma)k_2^2}. \qquad (112)$$

From this expression, the asymptotic form of $C_0(\boldsymbol{r})$ in the range $|\boldsymbol{r}| \gg \ell_{\rm int}$ is derived as

$$C_0(\boldsymbol{r}) \simeq -\frac{\lambda\bar{\rho}^2\hat{u}(0)\delta}{2\pi T\sqrt{[g_\delta(0)g_0(0)]^3 DT/\gamma}} \times \frac{[g_\delta(0)D]^{-1}r_1^2 - [g_0(0)(T/\gamma)]^{-1}r_2^2}{\{[g_\delta(0)D]^{-1}r_1^2 + [g_0(0)(T/\gamma)]^{-1}r_2^2\}^2}, \qquad (113)$$

where $\boldsymbol{r}=(r_1,r_2)$. This represents a long-range correlation of the type $1/r^2$.

2. Case (ii)

Next, let us consider the case $\ell_{\rm int} \ll \ell$. In this case, we obtain the continuity equation for ψ with the current $\mathcal{J}_\alpha$ expressed as

$$\mathcal{J}_1(\boldsymbol{x},t) = (v_{\rm s} + 2\bar{\rho}\lambda\nu)\psi - D\frac{\partial\psi}{\partial x_1} + \sqrt{\bar{\rho}+\psi(\boldsymbol{x},t)}\,\Xi_1(\boldsymbol{x},t) + \lambda\nu\psi^2(\boldsymbol{x},t),$$

$$\mathcal{J}_2(\boldsymbol{x},t) = -\frac{T}{\gamma}\frac{\partial\psi}{\partial x_2} + \sqrt{\bar{\rho}+\psi(\boldsymbol{x},t)}\,\Xi_2(\boldsymbol{x},t). \qquad (114)$$

Further, for simplicity, we assume that the ψ dependence of the noise term can be neglected. Then, the evolution equation coincides with the special case of that investigated in Ref. [10], where the fluctuation-dissipation relation of the second kind is satisfied. According to the result of Ref. [10], the equal-time correlation function is not modified by the interaction effects up to the second order of λ in this case [see Eq. (B2) in Ref. [10]]. Thus, we conclude that there is no long-range correlation within this approximation.

IV. CONCLUDING REMARK

The main achievement of this study is the derivation of the coarse-grained fluctuating hydrodynamic equation for the driven many-body Langevin system. In the two asymptotic cases for the interaction range between particles, which are given by Eqs. (75) and (89), we derived the two expressions of particle current, Eqs. (87) and (96), respectively, with the continuity equation expressed by Eq. (65). Using the obtained evolution equations, we calculated the equal-time correlation function of the coarse-grained density field for each case. We found that this correlation function exhibits the long-range correlation of the type r^{-d} in the case given by Eq. (75), while no such behavior was observed in the case given by Eq. (89).

In order to understand the connection between the two qualitatively different behaviors, we need to calculate the correlation function without focusing on the limiting cases. Although we can construct the function $\bar{u}(\theta,\boldsymbol{X})$ from an arbitrary interaction potential $u(\boldsymbol{x})$ by means of the method developed in Appendix B, it was not easy to develop a systematic and generally applied perturbation expansion that leads to the correlation function. The problem will be studied in the future.

We derived the coarse-grained fluctuating hydrodynamic equation by applying a singular perturbation method to a stochastic partial differential equation. The method is standard except for the treatment of space-time noise (see Appendix B). We expect that our method can be used to investigate other related problems such as phase diffusion behavior in periodic pattern formations under the influence of noise. We also note that the derived coarse-grained fluctuating hydrodynamic equations in this study contain nonlinear functions of f such as $v_{\rm s}$, D, B_α, δ, and ν. Therefore, we can discuss the system behavior in a nonlinear range with respect to f.

The long-range correlation we obtained for the case given by Eq. (75) has the essentially same mechanism as that of Ref. [20] which studied the system consisting of two Brownian particles under an external force. Thus, the result in this paper is regarded as an extension of that in Ref. [20], although the Fokker-Plank equation was analyzed in this reference. On the other hand, the long-distance behavior for the case given by Eq. (89) might be strange, because it has been speculated that an anisotropic system without the detailed balance condition generically exhibits the long-range correlation [8]. Here, it is noteworthy that the long-range correlation of the type r^{-d} does not appear in driven lattice gases with the evolution rule called an exponential method, while it appears in the cases of a heat bath method and a Metropolis method [11]. It might be interesting to find a connection of our result with that reported in Ref. [11].

Our calculation result for the correlation function given by Eq. (111) provides the functional form of its short-range part. In contrast to the long-range part, the short-range part depends on the details of the system such as the selection of the interaction potential. Such a nonuniversal part has never been investigated intensively. Here, let us recall that the statistical properties of density fluctuations are described by the free energy function of the system if the system is in equilibrium. Therefore, it might be expected that the short-range part of the correlation function is related to a thermodynamic function extended to nonequilibrium steady states. Although thermodynamics in nonequilibrium steady states has not yet been established, there exists one promising approach to construct a consistent framework whose validity can be checked experimentally [13]. According to this framework, the statistical properties of density fluctuations can be described by an extended free energy determined operationally when the effect of the long-range correlation is removed. Indeed, by numerical experiments on a driven lattice gas, it was demonstrated that the intensity of density fluctuations of a particular type is determined by an extended free energy [12], and this was proved in Ref. [13]. We expect that a similar analysis can be performed for the Langevin system under investigation in this study. The calculation of the short-range part of the correlation function is indispensable in this analysis.

The most ambitious goal is to provide a unified description of density fluctuations in an elegant manner. Even if the short-range part of the statistical property of density fluctuations is determined by an extended thermodynamic function, the long-range correlation that destroys the extensive nature of the system is obviously out of thermodynamic consideration. Here, it should be noted that a variational principle referred to as *the additivity principle* is effective to describe the long-range behavior of density fluctuations in nonequilibrium lattice gases [21,22]. Although we do not know a class of models to which this principle can be applied, it is interesting to determine whether this principle can be applied to the Langevin system under investigation.

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APPENDIX A: FLUCTUATING HYDRODYNAMICS FOR A FINE-GRAINED DENSITY FIELD

We derive the evolution equation for a fine-grained density field from the Langevin equation given by Eq. (1). Mathematically, the essence of the derivation is in employing the Ito formula for arbitrary functions of the fluctuating variable [23], and the evolution equation for the fine-grained density field is obtained by a standard treatment of Dirac's δ function. This approach was performed in Ref. [14]. Although Ref. [14] provides sufficient information on the derivation of the evolution equation for a fine-grained density field, we present a different method for the derivation in this appendix. We find that this method is less mathematical, but more pedagogical than the standard one.

Let Δt be a sufficiently small time interval, and set $t_n = n\Delta t$. Then, from Eq. (1), the movement of each particle during the time interval, $\Delta x_{i\alpha}(t_n) \equiv x_{i\alpha}(t_{n+1}) - x_{i\alpha}(t_n)$, can be expressed as

$$\gamma \Delta x_{i\alpha}(t_n) = \left[\left(f - \frac{\partial U(x_{i1})}{\partial x_{i1}}\right)\delta_{1\alpha} - \sum_{j\neq i}\frac{\partial u(\boldsymbol{x}_i - \boldsymbol{x}_j)}{\partial x_{i\alpha}}\right]_{[\boldsymbol{x}_\ell = \boldsymbol{x}_\ell(t_n)]}\Delta t + \hat{W}_{i\alpha}(t_n) + O((\Delta t)^{3/2}), \tag{A1}$$

where

$$\hat{W}_{i\alpha}(t_n) = \int_{t_n}^{t_{n+1}} dt\, R_{i\alpha}(t), \tag{A2}$$

and it should be noted that the equality

$$\hat{W}_{i\alpha}(t_n)\hat{W}_{j\beta}(t_m) = 2\gamma T \delta_{ij}\delta_{\alpha\beta}\delta_{nm}\Delta t \tag{A3}$$

holds almost surely [23].

Next, for the fine-grained density field $\rho_{\rm d}(\boldsymbol{x},t)$ defined by Eq. (3), we obtain

$$\gamma[\rho_{\rm d}(\boldsymbol{x},t_{n+1}) - \rho_{\rm d}(\boldsymbol{x},t_n)] = \sum_{i\alpha}\Delta x_{i\alpha}(t_n)\frac{\partial}{\partial x_{i\alpha}}\rho_{\rm d}(\boldsymbol{x},t_n) + \frac{1}{2}\sum_{i\alpha}\sum_{j\beta}\Delta x_{i\alpha}(t_n)\Delta x_{j\beta}(t_n) \times \frac{\partial}{\partial x_{i\alpha}}\frac{\partial}{\partial x_{j\beta}}\rho_{\rm d}(\boldsymbol{x},t_n) + O((\Delta t)^{3/2}). \tag{A4}$$

Substituting Eq. (A1) into the above expression and using Eq. (A3), we derive

$$\gamma[\rho_{\rm d}(\boldsymbol{x},t_{n+1}) - \rho_{\rm d}(\boldsymbol{x},t_n)] = -\Delta t\frac{\partial}{\partial x_1}\left[\left(f - \frac{\partial U(x_1)}{\partial x_1}\right)\rho_{\rm d}(\boldsymbol{x},t_n)\right] + \Delta t\frac{\partial}{\partial \boldsymbol{x}}\cdot\int d^2\boldsymbol{y}\, \rho_{\rm d}(\boldsymbol{x},t_n)\frac{\partial u(\boldsymbol{x}-\boldsymbol{y})}{\partial \boldsymbol{x}} \times \rho_{\rm d}(\boldsymbol{y},t_n) + T\Delta t\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right) \times \rho_{\rm d}(\boldsymbol{x},t_n) - \frac{\partial}{\partial \boldsymbol{x}}\cdot \boldsymbol{W}(\boldsymbol{x},t_n) + O((\Delta t)^{3/2}), \tag{A5}$$

where we have defined

$$W_\alpha(\boldsymbol{x},t_n) \equiv \sum_{i=1}^{N}\hat{W}_{i\alpha}(t_n)\delta^2(\boldsymbol{x}-\boldsymbol{x}_i). \tag{A6}$$

Note that $W_\alpha(\boldsymbol{x},t_n)$ satisfies

$$\langle W_\alpha(\boldsymbol{x},t_n)W_\beta(\boldsymbol{x}',t_m)\rangle = 2\gamma T\rho_{\rm d}(\boldsymbol{x},t_n)\delta_{\alpha\beta}\delta^2(\boldsymbol{x}-\boldsymbol{x}')\delta_{mn}\Delta t. \tag{A7}$$

Finally, taking the limit $\Delta t \to 0$, we obtain

$$\frac{\partial \rho_{\rm d}(\boldsymbol{x},t)}{\partial t} = -\frac{1}{\gamma}\frac{\partial}{\partial x_1}\left[\left(-\frac{\partial U(x_1)}{\partial x_1} + f\right)\rho_{\rm d}(\boldsymbol{x},t)\right] + \frac{1}{\gamma}\frac{\partial}{\partial \boldsymbol{x}}\int d^2\boldsymbol{y}\, \rho_{\rm d}(\boldsymbol{x},t)\frac{\partial u(\boldsymbol{x}-\boldsymbol{y})}{\partial \boldsymbol{x}}\rho_{\rm d}(\boldsymbol{y},t) + \frac{T}{\gamma}\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right)\rho_{\rm d}(\boldsymbol{x},t) - \frac{\partial}{\partial \boldsymbol{x}}\cdot\sqrt{\frac{T\rho_{\rm d}(\boldsymbol{x},t)}{\gamma}}\boldsymbol{\xi}(\boldsymbol{x},t), \tag{A8}$$

where $\xi_\alpha(\boldsymbol{x},t)$ satisfies Eq. (7). It is easily confirmed that the final expression is equivalent to Eq. (4) with Eq. (5).

APPENDIX B: FUNCTION OF (θ, X)

In Sec. III A, we introduced the functions $\bar{\boldsymbol{\xi}}(\theta,\boldsymbol{X},t)$ and $\bar{u}(\theta,\boldsymbol{X})$ corresponding to $\boldsymbol{\xi}(\boldsymbol{x},t)$ and $u(\boldsymbol{x})$, respectively. In this appendix, we present a method to construct the function of $(\theta,\boldsymbol{X})$. For simplicity, we consider functions defined in a one-dimensional interval, but the argument presented below can be extended to functions in two-or higher-dimensional regions.

Concretely, we construct a function $\bar{\phi}(\theta,X)$ corresponding to a function $\phi(x)$, where $\phi(x)$ is defined in the interval

$[0,L]$ and $\bar{\phi}(\theta,X)$ is defined in the region $[0,\ell)\times[0,\ell]$. We assume that there exists an integer N_1 satisfying $L=N_1\ell$. We explain a numerical computation method to obtain the function $\bar{\phi}(\theta,X)$ from $\phi(x)$ without a rigorous mathematical argument.

We first divide the interval $[0,L]$ into small segments $[ia,(i+1)a)$, where $0\leqslant i\leqslant N_0N_1-1$ and $a=\ell/N_0$. For the function $\phi(x)$, we set

$$\phi_i \equiv \phi(ia). \tag{B1}$$

Then, ϕ_i is regarded as a real-valued function from integers $0\leqslant i\leqslant N_0N_1-1$. For each i, there exists a unique pair of the integers i_0 and i_1 that satisfy

$$i = i_1N_0 + i_0, \tag{B2}$$

where $0\leqslant i_0\leqslant N_0-1$ and $0\leqslant i_1\leqslant N_1-1$. Hereinafter, i_0 and i_1 are regarded as functions of i. Using this notation, we define the function $\bar{\phi}_{i_0,i_1}$ as

$$\bar{\phi}_{i_0,i_1} = \phi_i. \tag{B3}$$

We expect that the function $\phi(x)$ appearing in physics is well approximated by using ϕ_i with a sufficiently small a. Then, for the function $\phi(x)$, we define $\bar{\phi}(\theta,X)$ by the relation

$$\bar{\phi}(\theta,X) = \bar{\phi}_{i_0,i_1}, \tag{B4}$$

with $\lfloor\theta/a\rfloor=i_0$ and $\lfloor XL/\ell^2-\theta/\ell\rfloor=i_1$. Here $\lfloor x\rfloor$ is the Gauss notation that represents the maximum integer less than x. (Mathematically speaking, we should argue the limit $a\rightarrow 0$ and a class of functions carefully, but this argument is not considered in this study)

Here, neglecting the irregularity arising from the Gauss notation, we write conventionally

$$X \simeq \frac{\ell}{L}(\ell i_1 + a i_0). \tag{B5}$$

Using Eq. (B2), this implies $X\simeq\epsilon x$. That is, the coordinate X thus defined is the large-scale coordinate describing the long-distance behavior. Next, we explain differentiation, integration, and noise for the functions of (θ,X).

Differentiation. Let $\phi(x)$ be a smooth function. The differentiation of $\phi(x)$ is approximated by $(\phi_{i+1}-\phi_i)/a$. We then have

$$\phi_{i+1} - \phi_i = \bar{\phi}_{i_0+1,i_1} - \bar{\phi}_{i_0,i_1} \tag{B6}$$

for $0\leqslant i_0\leqslant N_0-2$. From this, we derive

$$\phi(x+a) - \phi(x) \simeq \bar{\phi}\left(\theta + a, X + \frac{\ell}{L}a\right) - \bar{\phi}(\theta,X) \simeq \partial_\theta\bar{\phi}(\theta,X)a + \frac{\ell}{L}a\partial_X\bar{\phi}(\theta,X) + O(a^2), \tag{B7}$$

where the approximation in the first line originates from the discretization error and the fact that the irregularity of the Gauss notation was not considered. Further, in the second line, we have treated $\bar{\phi}(\theta,X)$ as a differentiable function that might be allowed in the appropriate limit $a\rightarrow 0$. From Eq. (B7), we obtain

$$\partial_x\phi(x) = (\partial_\theta + \epsilon\partial_X)\bar{\phi}(\theta,X). \tag{B8}$$

Integration. Let $\phi(x)$ be an integrable function. Then, we calculate

$$\begin{aligned}\int dx\, \phi(x) &\simeq a\sum_{i=0}^{N_0N_1-1}\phi_i = a\sum_{i_0=0}^{N_0-1}\sum_{i_1=0}^{N_1-1}\bar{\phi}_{i_0,i_1} \\ &\simeq a\sum_{i_0=0}^{N_0-1}\sum_{i_1=0}^{N_1-1}\bar{\phi}\left(i_0a, \frac{\ell}{L}(\ell i_1 + a i_0)\right) \\ &\simeq \int_0^\ell \frac{d\theta}{\ell}\int_0^\ell \frac{dX}{\epsilon}\bar{\phi}(\theta,X),\end{aligned} \tag{B9}$$

where the irregularity originating from the Gauss notation is not considered (the third line) and the limit $a\rightarrow 0$ and $\epsilon\rightarrow 0$ is taken (the forth line).

Coordinate-dependent noise. Let $\xi(x,t)$ be Gaussian white noise satisfying

$$\langle \xi(x,t)\xi(x',t')\rangle = 2\delta(x-x')\delta(t-t'). \tag{B10}$$

We set

$$\phi_i(t) = \frac{1}{a}\int_{ai}^{a(i+1)} dx\, \xi(x,t) \tag{B11}$$

with small a. For this discretized noise, we define $\bar{\phi}_{i_0,i_1}(t)$ and $\bar{\phi}(\theta,X,t)$ in the same manner as that for the case in which $\phi_i=\phi(ia)$. Conventionally, we denote $\bar{\phi}(\theta,X,t)$ as $\bar{\xi}(\theta,X,t)$ (see Sec. III A). Based on the definitions described above, we can derive

$$\langle \bar{\xi}(\theta,X,t)\bar{\xi}(\theta',X',t')\rangle = 2\epsilon\ell\,\delta(\theta-\theta')\delta(X-X')\delta(t-t'). \tag{B12}$$

Let $\varphi(\theta)$ be a smooth function that satisfies $\varphi(0)=\varphi(\ell)$. We denote the Stratonovich product of $\varphi(\theta)$ and $\bar{\xi}(\theta,X,t)$ as $\varphi(\theta)\circ\bar{\xi}(\theta,X,t)$. This product can be written by using the discretized form ϕ_{i_0,i_1} with an additional definition $\bar{\phi}_{N_0,i_1}=\bar{\phi}_{0,i_1}$. From this, we obtain

$$\int_0^\ell d\theta\, \partial_\theta[\varphi(\theta)\circ\bar{\xi}(\theta,X,t)] = 0. \tag{B13}$$

This formula is used for obtaining Eq. (53).

APPENDIX C: PROOF OF Eqs. (68)–(71)

In this appendix, we present the proofs of Eqs. (68)–(71). We first prove the key equality

$$\frac{d}{d\theta}\left(\frac{b(\theta)}{p_s(\theta)}\right) + 1 = \left(\int_0^\ell \frac{d\theta'}{\ell}I_-(\theta')\right)^{-1} I_+(\theta). \tag{C1}$$

All the equations can be derived from Eq. (C1).

1. Proof of the key equality

We first derive an explicit form of $b(\theta)$ defined in Eq. (61). Applying $\hat{M}^{(0)\dagger}$ given by Eq. (41) to both the left- and right-hand sides of Eq. (61), we obtain the following differential equation for $b(\theta)$:

$$\frac{d}{d\theta}\left(\frac{v_s}{p_s(\theta)}b(\theta)-\frac{T}{\gamma}\frac{d\ln p_s(\theta)}{d\theta}b(\theta)+\frac{T}{\gamma}\frac{db(\theta)}{d\theta}\right)=v_s[p_s(\theta)-1]-\frac{T}{\gamma}\frac{dp_s(\theta)}{d\theta}. \tag{C2}$$

Integrating Eq. (C2), we obtain the following first-order differential equation for b:

$$\frac{v_s}{p_s}b(\theta)-\frac{T}{\gamma}\frac{d\ln p_s(\theta)}{d\theta}b(\theta)+\frac{T}{\gamma}\frac{db(\theta)}{d\theta}=v_s[H(\theta)-\theta-K_1]-\frac{T}{\gamma}p_s(\theta), \tag{C3}$$

where K_1 is a constant whose value is determined later. Here, $H(\theta)$ and $V(\theta)$ are defined as

$$H(\theta)\equiv\int_0^\theta d\theta'\ p_s(\theta'), \tag{C4}$$

$$V(\theta)\equiv U(\theta)-f\theta. \tag{C5}$$

We introduce $\bar{b}$ in the equation

$$b(\theta)=p_s(\theta)[H(\theta)-\theta+\bar{b}(\theta)], \tag{C6}$$

and then rewrite Eq. (C3) as

$$v_s[\bar{b}(\theta)+K_1]+\frac{T}{\gamma}p_s(\theta)\left(p_s(\theta)+\frac{d\bar{b}(\theta)}{d\theta}\right)=0. \tag{C7}$$

Using the equality

$$v_s p_s e^{\beta V}=-\frac{T}{\gamma}p_s\frac{d}{d\theta}(p_s e^{\beta V}), \tag{C8}$$

we obtain the solution of Eq. (C7) as

$$\bar{b}(\theta)=-K_1+p_s(\theta)e^{\beta V(\theta)}[K_2-G(\theta)], \tag{C9}$$

where K_2 is a constant that is determined later and $G(\theta)$ is defined as

$$G(\theta)=\int_0^\theta d\theta'\ e^{-\beta V(\theta')}. \tag{C10}$$

Substituting Eq. (C9) into Eq. (C6), we write

$$\frac{b(\theta)}{p_s(\theta)}=H(\theta)-\theta-K_1+p_s(\theta)e^{\beta V(\theta)}[K_2-G(\theta)]. \tag{C11}$$

Now, K_1 and K_2 are determined from the conditions $(b,\Phi_0)=0$ and $b(0)=b(\ell)$. The results are as follows:

$$K_2=\frac{1}{1-e^{\beta f\ell}}G(\ell), \tag{C12}$$

$$K_1=\int_0^\ell\frac{d\theta}{\ell}\{p_s(\theta)[H(\theta)-\theta]+p_s^2(\theta)e^{\beta V(\theta)}[K_2-G(\theta)]\}. \tag{C13}$$

Next, we note the identity

$$\int_0^\ell d\theta'\ \phi(\theta')e^{\beta f\theta'}-(1-e^{\beta f\ell})\int_0^\theta d\theta'\ \phi(\theta')e^{\beta f\theta'}=e^{\beta f\theta}\int_0^\ell d\theta'\ \phi(\theta'+\theta)e^{\beta f\theta'} \tag{C14}$$

for an arbitrary periodic function $\phi(\theta)$ with period ℓ. Setting $\phi=e^{-\beta U}$ in Eq. (C14), we obtain

$$K_2-G(\theta)=\frac{1}{1-e^{\beta f\ell}}e^{-\beta V(\theta)}I_+(\theta). \tag{C15}$$

Then, substituting Eq. (C15) into Eq. (C11) and multiplying v_s by both the left- and right-hand sides, we obtain

$$v_s\frac{b(\theta)}{p_s(\theta)}=v_s[H(\theta)-\theta-K_1]-\frac{T}{\gamma}\left(\int_0^\ell\frac{d\theta'}{\ell}I_-(\theta')\right)^{-1}p_s(\theta)I_+(\theta), \tag{C16}$$

where we have used Eq. (15).

On the other hand, we rewrite Eq. (C3) as

$$\frac{T}{\gamma}p_s(\theta)\left[\frac{d}{d\theta}\left(\frac{b(\theta)}{p_s(\theta)}\right)+1\right]=-\frac{v_s}{p_s}b+v_s[H(\theta)-\theta-K_1]. \tag{C17}$$

Comparing Eqs. (C16) and (C17), we obtain Eq. (C1).

2. Proof of Eq. (70)

We can rewrite D in Eq. (58) as

$$D=v_s\int_0^\ell\frac{d\theta}{\ell}\frac{b(\theta)}{p_s(\theta)}+\frac{T}{\gamma}\int_0^\ell\frac{d\theta}{\ell}\left[\frac{d}{d\theta}\left(\frac{b(\theta)}{p_s(\theta)}\right)+1\right]p_s(\theta). \tag{C18}$$

Then, substituting Eqs. (C16) and (C1) into Eq. (C18), we obtain

$$D=v_s\int_0^\ell\frac{d\theta}{\ell}[H(\theta)-\theta]-v_sK_1. \tag{C19}$$

Substituting Eq. (C13) into Eq. (C19) and using Eq. (C15), we rewrite (C19) as

$$D=-v_s\int_0^\ell\frac{d\theta}{\ell}p_s^2(\theta)e^{\beta V(\theta)}\left(\frac{1}{e^{\beta f\ell}-1}e^{-\beta V(\theta)}I_+(\theta)\right), \tag{C20}$$

where we have used the identity

$$\int_0^\ell\frac{d\theta}{\ell}[p_s(\theta)-1][\theta-H(\theta)]=0. \tag{C21}$$

Finally, substituting Eqs. (15) and (16) into Eq. (C20), we obtain Eq. (70).

3. Proof of Eqs. (68), (69), and (71)

We first calculate the correlation function of $\eta_2(\boldsymbol{X},t)$ as follows:

$$\langle \eta_2(\boldsymbol{X},t)\eta_2(\boldsymbol{X}',t')\rangle = \frac{T}{\gamma}\int_0^\ell \frac{d\theta}{\ell}\int_0^\ell \frac{d\theta'}{\ell}\frac{\Psi_0(\theta)}{\sqrt{p_s(\theta)}}\frac{\Psi_0(\theta')}{\sqrt{p_s(\theta')}}\langle \bar{\xi}_2(\theta,\boldsymbol{X},t)\bar{\xi}_2(\theta',\boldsymbol{X}',t')\rangle = \frac{T}{\gamma}\int_0^\ell \frac{d\theta}{\ell}\int_0^\ell \frac{d\theta'}{\ell}\frac{\Psi_0(\theta)\Psi_0(\theta')}{\sqrt{p_s(\theta)p_s(\theta')}}2\ell\epsilon^2\delta^2(\boldsymbol{X}-\boldsymbol{X}')\delta(\theta - \theta')\delta(t-t') = \frac{2T}{\gamma}\epsilon^2\delta^2(\boldsymbol{X}-\boldsymbol{X}')\delta(t-t'), \tag{C22}$$

where we have used (B12). Using Eqs. (64) and (67), we obtain Eq. (69).

Next, we consider the correlation function of $\zeta(\boldsymbol{X},t)+\eta_1(\boldsymbol{X},t)$ as follows:

$$\langle[\zeta(\boldsymbol{X},t)+\eta_1(\boldsymbol{X},t)][\zeta(\boldsymbol{X}',t')+\eta_1(\boldsymbol{X}',t')]\rangle = \frac{T}{\gamma}\int_0^\ell \frac{d\theta}{\ell}\int_0^\ell \frac{d\theta'}{\ell}\left[\frac{d}{d\theta}\left(\frac{b(\theta)}{p_s(\theta)}\right)+1\right]\left[\frac{d}{d\theta'}\left(\frac{b(\theta')}{p_s(\theta')}\right)+1\right]\sqrt{p_s(\theta)p_s(\theta')} \times\langle\bar{\xi}_1(\theta,\boldsymbol{X},t)\bar{\xi}_1(\theta',\boldsymbol{X}',t')\rangle = \frac{2T}{\gamma}\int_0^\ell \frac{d\theta}{\ell}\left[\frac{d}{d\theta}\left(\frac{b(\theta)}{p_s(\theta)}\right)+1\right]^2 p_s(\theta)\epsilon^2\delta^2(\boldsymbol{X}-\boldsymbol{X}')\delta(t-t'). \tag{C23}$$

This corresponds to Eq. (68).

Finally, substituting Eqs. (C1) and (15) into Eq. (68), we obtain

$$B_{11} = \frac{T}{\gamma}\left(\int_0^\ell \frac{d\theta}{\ell}I_-(\theta)\right)^{-3}\int_0^\ell \frac{d\theta}{\ell}I_-(\theta)[I_+(\theta)]^2. \tag{C24}$$

Because the right-hand side of Eq. (C24) is invariant for the exchange of I_+ and I_- (see Ref. [18]), B_{11} in Eq. (C24) is equal to D in Eq. (70). This corresponds to Eq. (71).

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